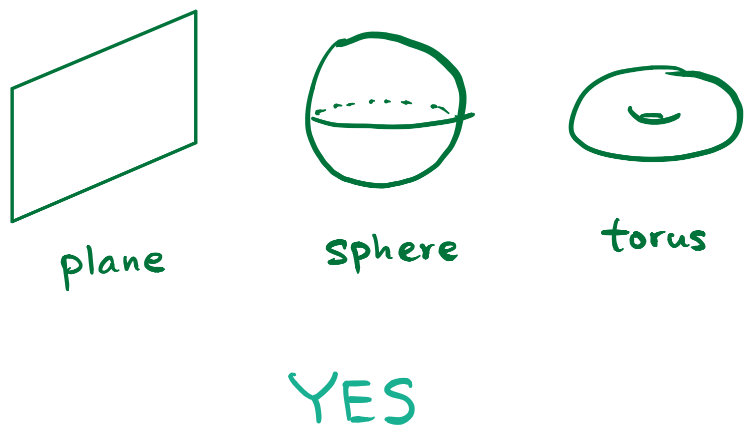
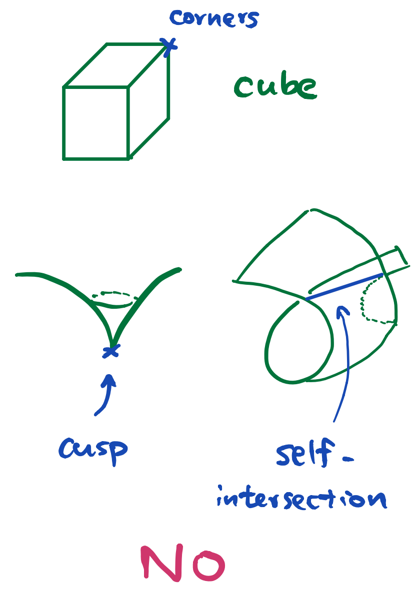
§ Surfaces in $\mathbb{R}^3$ (do Carmo § 2.2)

Q: Which one do we want to consider as "surfaces"?





Basic idea: A "surface" is an object that ① locally ② looks like a piece of $\mathbb{R}^2$.

Definition: A (regular) surface is a subset

$$S \subseteq \mathbb{R}^3$$

s.t. $\forall\, p \in S$, $\exists$ nbd. of p (in S) $V \subseteq S$

and a smooth map (called parametrization / chart)

$$\mathbf{X} : \underset{\text{open}}{U \subseteq \mathbb{R}^2} \longrightarrow V$$

s.t. (1) $\mathbf{X} : U \to V$ is a homeomorphism.

* (2) The differential $d\mathbf{X}|_q$ is 1-1 $\forall\, q \in U$.

Explanation on *: More explicitly,

$$\mathbf{X}(u,v) = \big(\underbrace{x(u,v)}, \underbrace{y(u,v)}, \underbrace{z(u,v)}\big), \quad q=(u,v)\in\mathcal{U}$$

smooth functions

The differential of $\mathbf{X}$ at $q\in\mathcal{U}$ is a linear map

$$d\mathbf{X}\big|_q : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

which can be expressed in matrix form (w.r.t. std basis)

$$d\mathbf{X}\big|_q = \left.\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}\right|_q =: \left.\begin{pmatrix} | & | \\ \frac{\partial \mathbf{X}}{\partial u} & \frac{\partial \mathbf{X}}{\partial v} \\ | & | \end{pmatrix}\right|_q$$

(*) $\boxed{d\mathbf{X}\big|_q \text{ is 1-1} \iff \frac{\partial \mathbf{X}}{\partial u}, \frac{\partial \mathbf{X}}{\partial v} \text{ are linearly independent.}}$

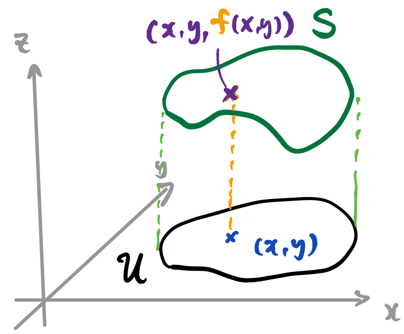
Notation: We will write $\mathbf{X}_u$, $\mathbf{X}_v$ to denote $\frac{\partial \mathbf{X}}{\partial u}$, $\frac{\partial \mathbf{X}}{\partial v}$ respectively.

Remark: (*) is equivalent to any one of the following:

- $\mathbf{X}_u \times \mathbf{X}_v \neq \vec{0}$ at q
- $\text{rank}(d\mathbf{X}) = 2$ at q
- $\exists$ 2×2 sub-matrix of $d\mathbf{X}\big|_q = \left.\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}\right|_q$ which is invertible.

Example 1 : Graphical surfaces

Given a smooth function $f : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$



$$S = \text{graph}(f) = \{ z = f(x,y) \mid (x,y) \in \mathcal{U} \}$$

is a (regular) surface.

Why? Consider the smooth map

$$\begin{aligned} \mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^2 &\longrightarrow S \subseteq \mathbb{R}^3 \\ (u,v) &\longmapsto (u, v, f(u,v)) \end{aligned}$$

Clearly, $\mathbf{X} : \mathcal{U} \to S$ is a homeomorphism.

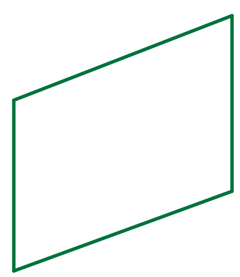
$$\left.\begin{aligned} \mathbf{X}_u := \frac{\partial \mathbf{X}}{\partial u} &= \left(1, 0, \frac{\partial f}{\partial u}\right) \\ \mathbf{X}_v := \frac{\partial \mathbf{X}}{\partial v} &= \left(0, 1, \frac{\partial f}{\partial v}\right) \end{aligned}\right\} \text{ always linearly independent.}$$

Alternatively,

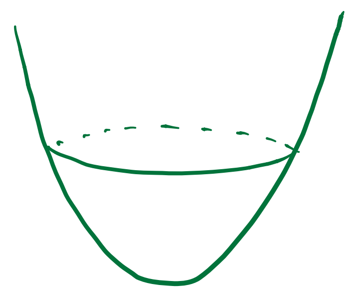
$$\mathbf{X}_u \times \mathbf{X}_v = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1 \right) \neq \vec{0}$$

Examples of graphical surfaces

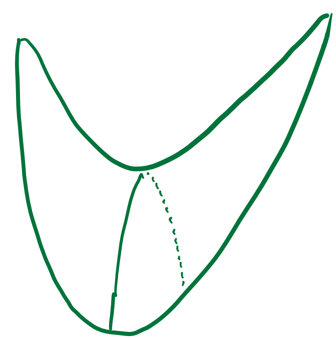
- $f(x,y) = ax + by + c \implies$ planes
- $f(x,y) = x^2 + y^2 \implies$ paraboloid
- $f(x,y) = x^2 - y^2 \implies$ hyperboloid



plane



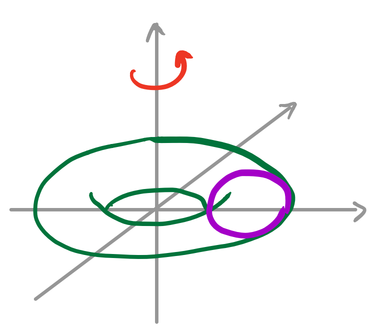
paraboloid



hyperboloid

Note: The entire surface can be covered by 1 chart.

Example 2 : Torus of revolution



$$S = \{ (\sqrt{x^2+y^2} - a)^2 + z^2 = r^2 \}$$

where $a > r > 0$ are constants.

is a surface.

Define $\mathbf{X} : (0,2\pi) \times (0,2\pi) = \mathcal{U} \longrightarrow \mathbb{R}^3$ by

$$\mathbf{X}(u,v) = \Big((a + r\cos u)\cos v \, , \, (a + r\cos u)\sin v \, , \, r\sin u \Big)$$

Taking partial derivatives w.r.t. u and v,

$$X_u = r(-\sin u \cos v, -\sin u \sin v, \cos u)$$

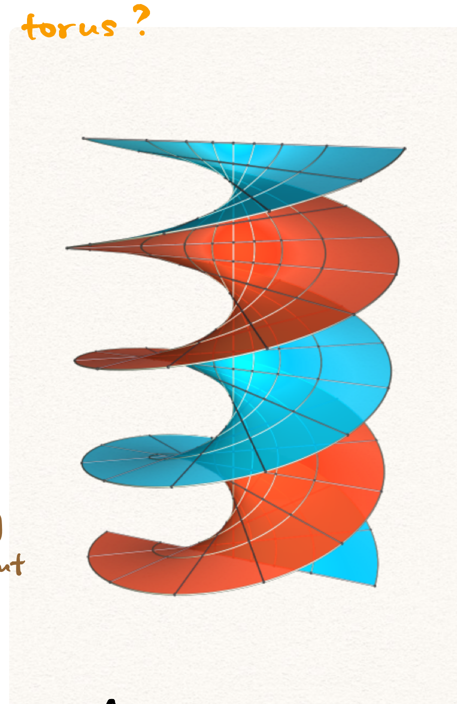
$$X_v = (a + r\cos u)(-\sin v, \cos v, 0)$$

$$X_u \times X_v = r(a + r\cos u)\underbrace{(-\cos u \cos v, -\cos u \sin v, -\sin u)}_{\neq \vec{0}}$$

So X_u & X_v are linearly independent.

Exercise: How many parametrizations do we need to cover the whole torus?

Example 3: Helicoid



Consider $X : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$X(u,v) = (v\cos u, v\sin u, u)$$

$$\left.\begin{aligned} X_u &= (-v\sin u, v\cos u, 1) \\ X_v &= (\cos u, \sin u, 0) \end{aligned}\right\} \text{linearly independent}$$

Therefore, the image $S := X(\mathbb{R}^2)$ is a surface.

Example 4: Ruled Surfaces

Let $\alpha : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$ be a regular space curve,

$\beta : I \longrightarrow \mathbb{R}^3$ be a smooth map s.t. $\beta \neq \vec{0}$ everywhere

Define $\mathbf{X} : I \times \mathbb{R} \to \mathbb{R}^3$ be

$$\mathbf{X}(u,v) = \alpha(u) + v\,\beta(u)$$

Q: Does $\mathbf{X}$ give a parametrization of a surface?

$$\mathbf{X}_u = \alpha'(u) + v\,\beta'(u)$$

$$\mathbf{X}_v = \beta(u)$$

$$\Rightarrow \quad \mathbf{X}_u \times \mathbf{X}_v = \underbrace{\alpha'(u) \times \beta(u) + v\,(\beta'(u) \times \beta(u))}_{\text{Need: } \neq \vec{0}.}$$

Examples include:

- generalized cylinder

  ($\beta \equiv$ constant vector)

- cones

  ($\beta = -\alpha$)